A contribution to the mathematical analysis of variable spindle speed machining

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It has been observed that the critical depth of cut in single-point machining can be increased by continuously varying the spindle speed.

Several more experimentally oriented analyses have confirmed this observation. Theoretical models have been set up but in general do not confirm the experimentally obtained results.

The analysis presented here, although starting from a different point of view, confirms some of the work by Sexton *et al.* It further yields detailed supplementary information, particularly concerning the nature of the chatter vibration under variable spindle speed and the dependence of the process on the amplitude of the speed variation. These results are illustrated by some specific experimental evidence.

Introduction

In recent years, several studies have been devoted to the investigation of variable spindle speed machining as a possible means to control the onset of 'chatter', an instability occurring in machine tools when the metal removal rates are increased beyond a critical value. Specifically it has been suggested both by experimental work[1,2] and by analogue computer simulation[3] that a (usually rather modest) increase of stability of the cutting process can be obtained when the spindle speed is continuously varied, thus allowing of a larger depth (or width) of cut; but the adverse effect has been observed as well.[2] These ambiguous results indicate the need for a theoretical understanding of the cutting process under variable spindle speed, if it is ever to acquire more than marginal applicability.

In a first attempt to build an adequate theory, Inamura and Sata[4] vastly overrated the effectiveness of spindle speed variation. The source of their error became apparent in the analysis by Sexton *et al.*,[5] whose estimates are far more realistic. Still these latter results are not completely satisfactory, because of reasons explained below (and partly pointed out by the authors[5] themselves); moreover full mathematical clarity is not always achieved in their treatment.

The analysis presented here, although starting from a rather different point of view, does confirm some of Sexton's assumptions, while providing them with a sound basis. More importantly, however, it yields some detailed supplementary information, in particular concerning the nature of the chatter vibration under variable spindle speed, and the dependence of the process on the amplitude of the speed variation (for small values of the latter). These results are illustrated by some specific experiments, mentioned at the end of this paper.

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An eigenvalue problem

It is generally agreed[3–5] that, under the assumptions that the machine tool structure has essentially one degree of freedom and a single predominant mode of vibration, a reasonable mathematical description of the cutting process under variable spindle speed is yielded by the following differential equation:

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2 x(t) = K[x(t-\phi(t)) - x(t)] \quad (1)$$

here $\omega_0/2\pi$ and ζ are the natural frequency respectively the damping ratio of the machine tool, K is proportional to the depth (or width) of cut, and ϕ is the time-lag function, i.e. $\phi(t)$ is the time elapsed at the moment t since the workpiece was at the same angular position one revolution earlier. The unknown real function x describes the displacement of the tool from its reference position. For additional information see reference 5.

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The problem consists in finding the least nonzero real value of K for which equation (1) admits of a vibrational solution x, which corresponds to the critical depth of cut where chatter occurs. The notion of 'vibrational solution' will be specified later on.

We are interested in the cases where the time-lag function ϕ assumes one of the forms:

$$\phi_c(t) = \tau \quad \text{(constant lag)} \tag{2}$$

or

$$\phi_m(t) = \tau + T \sin \omega_m t \quad \text{(sinusoidal lag modulation)} \tag{3}$$

The first case describes the cutting process where the workpiece rotates with a constant period τ; in the second case the time lag is modulated around a mean value τ, with an amplitude T and a frequency $\omega_m/2\pi$. Let us call K_c and $K_m(T)$ the corresponding least nonzero real values of K, thus corresponding to the critical depth of cut in the two cases. The ratio $K_m(T)/K_c$ gives the relative efficiency of cutting under variable spindle speed (with amplitude T of lag modulation) as compared with constant speed machining.

A few comments are in order to justify the choice of the function ϕ_m above. In the first place, a sinusoidal modulation of the time-lag (which is easier to treat mathematically) can be expected to have approximately the same effect as a sinusoidal modulation of the spindle speed (which is easier to realize in practice).[5] Secondly, all forms of modulation seem to be roughly equivalent,[1] with the possible exception of a rectangular signal.[3] The sinusoidal variation is chosen for mathematical convenience. Finally, the phase of ϕ has no influence on the value of K in equation (1). Specifically, if x_1 is a solution of equation (1) for $\phi = \phi_1$ and a particular value of K, then $x_2(t) = x_1(t - \theta)$ is a solution of (1) for $\phi(t) = \phi_1(t - \theta)$ and for the same K. Hence it is unnecessary to consider:

$$\phi(t) = \tau + T \sin(\omega_m t - \theta)$$

for all values of θ. This seems to have been overlooked by Sexton *et al.*[5] In particular it follows from the last remark that $K_m(T)$ does not change when T is replaced by $-T$ in ϕ_m, because such a replacement amounts to a phase shift over π/ω_m. In other words, $K_m(T)$ is an even function of T. This simple observation will lead to a substantial short cut in the sequel (see next section).

Let us now give a precise meaning to the notion of a 'vibrational solution' of equation (1). Following Sexton's interpretation[5] we look for an almost periodic solution of (1), or rather for an approximate solution. However, both the analogue computer simulation[3] and the mathematical analysis show that the solution cannot in general be expected to be periodic. Hence trying to find a strictly periodic approximate solution,[4,5] must lead to serious inaccuracies, as is indeed hinted at by Sexton *et al.* in their discussion.[5] To avoid this difficulty, which constitutes our main criticism of Sexton *et al.*,[5] we require that the approximate solution x be of the more general form:

$$x(t) = \mathrm{Re}\left(\sum_{k=1}^{n} a_k e^{j\alpha_k t}\right)$$

where the α_k are real and the a_k complex. In other words, x is (the real part of) a trigonometric polynomial.

The space $\mathscr{P}$ of trigonometric polynomials being uniformly dense in the space $\mathscr{H}$ of almost periodic functions,[6] we can hope that our approximate solution comes arbitrary close to the exact solution in the sense of the uniform norm, which is quite satisfactory (for material about almost periodic functions, we refer to the still very readable treatise[6] of the founder of the theory).

Next we proceed to reformulate equation (1) as an eigenvalue problem in the space $\mathscr{H}$. Define linear operators M, L_0 and $\tilde{L}$ in $\mathscr{H}$ by:

$$M = \left(\frac{\mathrm{d}}{\mathrm{d}t^2} + 2\zeta\omega_0 \frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right)^{-1} \tag{4}$$

$$(L_0 x)(t) = x(t - \tau) - x(t) \tag{5}$$

$$(\tilde{L}x)(t) = x(t - \tau - T \sin \omega_m t) - x(t) \tag{6}$$

Then equation (1) is equivalent to:

$$ML_0 x = \lambda x \tag{7}$$

for $\phi = \phi_c$ (2), if we put $\lambda = 1/K$, and with:

$$M\tilde{L}x = \tilde{\lambda} x \tag{8}$$

for $\phi = \phi_m$ (3), if we put $\tilde{\lambda} = 1/K$ in this case. Note that we have previously excluded the trivial case $K = 0$.

Let us introduce the notation $e_\alpha(t) = e^{j\alpha t}$ for all real α. The vectors $\{e_\alpha\}_{\alpha \in \mathbb{R}}$ form a basis for $\mathscr{H}$ in the Banach space sense and for $\mathscr{P}$ in the ordinary vector space sense. With respect to this basis, the operators M and L_0 are diagonal: it is easily checked from (4) and (5) that:

$$Me_\alpha = (\omega_0^2 - \alpha^2 + 2j\zeta\omega_0\alpha)^{-1} e_\alpha \tag{9}$$

$$L_0 e_\alpha = (e^{-j\alpha\tau} - 1) e_\alpha \tag{10}$$

Consequently the eigenvalue problem (7) is easily solved (in the complex plane): each e_α is an eigenvector with (possibly complex) eigenvalue:

$$\lambda_\alpha = \frac{e^{-j\alpha\tau} - 1}{\omega_0^2 - \alpha^2 + 2j\zeta\omega_0\alpha} \tag{11}$$

The problem of finding the largest real value $\lambda_c = 1/K_c$ among these eigenvalues is considered elsewhere.[5] The corresponding frequency $\alpha_0/2\pi$ (defined by $\lambda_{\alpha_0} = \lambda_c$) turns out to be near to the natural frequency $\omega_0/2\pi$ (at least if ζ is small) and the corresponding solution ('chatter'), is a pure vibration, of the form $x = \mathrm{Re}(ae_{\alpha_0})$.

In order to express the operator $\tilde{L}$ in the basis $\{e_\alpha\}$, we write the Taylor expansion:

$$\begin{aligned} &x(t - \tau + h) - x(t - \tau) \\ &\quad = h\frac{\mathrm{d}}{\mathrm{d}t}x(t - \tau) + \frac{h^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t - \tau) \\ &\quad + \frac{h^3}{6}\frac{\mathrm{d}^3}{\mathrm{d}t^3}x(t - \tau) + \frac{h^4}{24}\frac{\mathrm{d}^4}{\mathrm{d}t^4}(t - \tau) + \ldots \end{aligned}$$

of a solution x around $t - \tau$. Upon substituting $h = -T \sin \omega_m t$, we obtain:

$$\begin{aligned} &x(t - \tau - T \sin \omega_m t) - x(t - \tau) \\ &\quad = -T \sin \omega_m t \frac{\mathrm{d}}{\mathrm{d}t}x(t - \tau) \\ &\quad + \frac{T^2}{2}\sin^2 \omega_m t \frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t - \tau) \\ &\quad - \frac{T^3}{6}\sin^3 \omega_m t \frac{\mathrm{d}^3}{\mathrm{d}t^3}x(t - \tau) \\ &\quad + \frac{T^4}{24}\sin^4 \omega_m t \frac{\mathrm{d}^4}{\mathrm{d}t^4}x(t - \tau) - \ldots \end{aligned}$$

Hence we can develop $\tilde{L}$ in a series of operators:

$$\tilde{L} = L_0 + \epsilon L_1 + \epsilon^2 L_2 + \epsilon^3 L_3 + \epsilon^4 L_4 + \ldots$$

where $\epsilon = T/\tau$ is a nondimensional parameter and:

$$(L_1 x)(t) = -\tau \sin\omega_m t \frac{\mathrm{d}}{\mathrm{d}t} x(t-\tau)$$

$$(L_2 x)(t) = \frac{\tau^2}{2} \sin^2\omega_m t \frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t-\tau)$$

$$(L_3 x)(t) = -\frac{\tau^3}{6} \sin^3\omega_m t \frac{\mathrm{d}^3}{\mathrm{d}t^3} x(t-\tau)$$

$$(L_4 x)(t) = \frac{\tau^4}{24} \sin^4\omega_m t \frac{\mathrm{d}^4}{\mathrm{d}t^4} x(t-\tau) \ldots \text{etc.}$$

So we replace (8) by:

$$M\left(\sum_{k=0}^{\infty} \epsilon^k L_k\right) x = \tilde{\lambda} x \qquad (12)$$

an approximate solution of which will be given in the next section using standard techniques. On the other hand, it is a matter of straightforward calculation to obtain:

$$L_1 e_\alpha = \frac{\alpha\tau}{2} e^{-j\alpha\tau}(e_{\alpha-\omega_m} - e_{\alpha+\omega_m}) \qquad (13)$$

$$L_2 e_\alpha = \frac{(\alpha\tau)^2}{8} e^{-j\alpha\tau}(e_{\alpha-2\omega_m} - 2e_\alpha + e_{\alpha+2\omega_m}) \qquad (14)$$

$$L_3 e_\alpha = \frac{(\alpha\tau)^3}{48} e^{-j\alpha\tau}(e_{\alpha-3\omega_m} - 3e_{\alpha-\omega_m} + 3e_{\alpha+\omega_m} - e_{\alpha+3\omega_m}) \qquad (15)$$

$$L_4 e_\alpha = \frac{(\alpha\tau)^4}{384} e^{-j\alpha\tau}(e_{\alpha-4\omega_m} - 4e_{\alpha-2\omega_m} + 6e_\alpha - 4e_{\alpha+2\omega_m} + e_{\alpha+4\omega_m}) \qquad (16)$$

etc.

It is worthwhile noting that there is another way to obtain the formulae (13)-(16), which, although much less elementary, is interesting because it exhibits a certain relationship with the 'projection' method of Sexton *et al.*[5] First one writes down the Fourier expansion:

$$e^{-j\alpha T\sin\omega_m t} = \sum_{n=-\infty}^{+\infty} J_n(\alpha T)\, e^{jn\omega_m t}$$

using formula 9.1.22 in reference 7 for the Bessel functions J_n. This leads to:

$$e^{j\alpha(t-\tau-T\sin\omega_m t)} - e^{j\alpha\tau}$$
$$= (e^{-j\alpha\tau}J_0(\alpha T) - 1)\, e^{j\alpha t} + e^{-j\alpha\tau}\sum_{n=1}^{+\infty} J_n(\alpha T)$$
$$\times (e^{j(\alpha+n\omega_m t)} + (-1)^n e^{j(\alpha-n\omega_m t)})$$

since $J_{-n}(\alpha T) = (-1)^n J_n(\alpha T)$. The formulae (13)-(16) are then obtained using the series expansion:

$$J_n(\alpha T) = (\tfrac{1}{2}\alpha T)^n \sum_{k=0}^{+\infty} \frac{(-\frac{1}{4}\alpha^2 T^2)^k}{k!(n+k)!} \qquad (n \geqslant 0)$$

which can be found in reference 7 (9.1.10).;

A perturbation calculus

Observing that the parameter $\epsilon = T/\tau$, introduced in the previous section, is small (typically between 0.1 and 0.2) we consider the system (12) as a perturbation of the eigenvalue problem (7) and we construct an approximate solution of (12) using the well known techniques of perturbation calculus, e.g. reference 8.

Specifically, we assume that, for every real α, there exists an eigenvector $\tilde{e}(\alpha, \epsilon)$ of (12), with corresponding eigenvalue $\tilde{\lambda}(\alpha, \epsilon)$ such that $\tilde{e}(\alpha, 0) = e_\alpha$ and $\tilde{\lambda}(\alpha, 0) = \lambda_\alpha$. Moreover, we suppose that both $\tilde{e}(\alpha, \epsilon)$ and $\tilde{\lambda}(\alpha, \epsilon)$ admit of a series expansion in ϵ:

$$\tilde{e}(\alpha, \epsilon) = e_\alpha + \epsilon e_1(\alpha) + \epsilon^2 e_2(\alpha) + \epsilon^3 e_3(\alpha) + \epsilon^4 e_4(\alpha) + \ldots \qquad (17)$$

$$\tilde{\lambda}(\alpha, \epsilon) = \lambda_\alpha + \epsilon^2\lambda_2(\alpha) + \epsilon^4\lambda_4(\alpha) + \ldots \qquad (18)$$

where the $e_k(\alpha)$ are trigonometric polynomials and the $\lambda_{2k}(\alpha)$ are complex numbers. Notice that $\tilde{\lambda}(\alpha, \epsilon)$ is an even function of ϵ because $K_m(T)$ is an even function of T, as shown earlier in the paper.

Substituting (17) and (18) in (12) (with $x = \tilde{e}(\alpha, \epsilon)$) and equating the coefficients of the corresponding powers of ϵ, we obtain the following system of equations:

$ML_0 e_\alpha = \lambda_\alpha e_\alpha$, which is satisfied by definition,

$$ML_0 e_1(\alpha) + ML_1 e_\alpha = \lambda_\alpha e_1(\alpha) \qquad (19)$$

$$ML_0 e_2(\alpha) + ML_1 e_1(\alpha) + ML_2 e_\alpha = \lambda_\alpha e_2(\alpha) + \lambda_2(\alpha) e_\alpha$$

$$ML_0 e_3(\alpha) + ML_1 e_2(\alpha) + ML_2 e_1(\alpha) + ML_3 e_\alpha \qquad (20)$$
$$= \lambda_\alpha e_3(\alpha) + \lambda_2(\alpha)\, e_1(\alpha) \qquad (21)$$

$$ML_0 e_4(\alpha) + ML_1 e_3(\alpha) + ML_2 e_2(\alpha) + ML_3 e_1(\alpha) + ML_4 e_\alpha$$
$$= \lambda_\alpha e_4(\alpha) + \lambda_2(\alpha)\, e_2(\alpha) + \lambda_4(\alpha)\, e_\alpha \qquad (22)$$

etc.

It will be shown in the next section that these equations allow us to determine successively $\lambda_2(\alpha)$, $\lambda_4(\alpha)$, etc., and that $e_k(\alpha)$ can also be computed, at least up to an unimportant ambiguity; moreover they turn out to be trigonometric polynomials as required. But let us first indicate how the results of these computation can be used. By breaking off the series (18) after $n + 1$ terms, for all real α we obtain a (complex) approximate eigenvalue:

$$\tilde{\lambda} \simeq \lambda_\alpha + \epsilon^2\lambda_2(\alpha) + \epsilon^4\lambda_4(\alpha) + \ldots + \epsilon^{2n}\lambda_{2n}(\alpha) \qquad (23)$$

for equation (8) (or (12)), valid for suitably small values of ϵ (or T), according to the size of n. We are interested in the largest possible real value of $\tilde{\lambda}$ in (8), which we denote by $\lambda_m(T) = 1/K_m(T)$ to stress its dependence on T. When T is small, this value is likely to occur for α in (23) near to α_0 (recall that $\alpha_0/2\pi$ is the frequency of chatter at constant speed), hence also near to ω_0. Consequently we have to solve the equation:

$$\mathrm{Im}(\lambda_\alpha + \epsilon^2\lambda_2(\alpha) + \epsilon^4\lambda_4(\alpha) + \ldots + \epsilon^{2n}\lambda_{2n}(\alpha)) = 0 \qquad (24)$$

in the unknown real variable ϵ, for values of α close to α_0. Let then ϵ_1 be such a solution. According to the previous reasoning, the corresponding value of (23) is a reasonable estimate of the largest possible real value of $\tilde{\lambda}$ in (8) for $T_1 = \epsilon_1\tau$, so we put:

$$\lambda_m(T_1) = \lambda_\alpha + \epsilon_1^2\lambda_2(\alpha) + \epsilon_1^4\lambda_4(\alpha) + \ldots + \epsilon_1^{2n}\lambda_{2n}(\alpha) \qquad (25)$$

provided this expression is positive. For every value of α, we obtain in this way one or several points of the graph of $(T, K_m(T)/K_c)$. An example is given in *Figure 3*, which will be discussed later in the paper.

Explicit formulae

Here we give an explicit solution for equations (19)-(22). Throughout this section α has a fixed value and for typographical reasons we introduce new notations $ML_0 = A$, $ML_1 = B$, $ML_2 = C$, $e_1(\alpha) = v$, $e_2(\alpha) = w$, $e_3(\alpha) = z$, $\lambda_2(\alpha) = \mu$ and $\lambda_4(\alpha) = \nu$. It is clear from the equations (13)-(16) that all the $e_k(\alpha)$ must be linear combinations of the pure vibrations e_α, $e_{\alpha \pm \omega_m}$, $e_{\alpha \pm 2\omega_m}$, etc. So it is useful to define (for $k = 0, \pm 1, \pm 2, \ldots$):

$$u_k = e_{\alpha + k\omega_m} \quad \text{(in particular } u_0 = e_\alpha)$$
$$m(k) = (\omega_0^2 - (\alpha + k\omega_m)^2 + 2j\zeta\omega_0(\alpha + k\omega_m))^{-1}$$
$$\lambda(k) = \lambda_{\alpha + k\omega_m} = (e^{-j(\alpha + k\omega_m)\tau} - 1)\, m(k)$$
$$\text{(in particular } \lambda(0) = \lambda_\alpha)$$

With this new notation, equation (19) then takes the form:

$$Av + Bu_0 = \lambda(0)\, v \tag{26}$$

We want to compute v, which, as noticed earlier, is a linear combination of the u_k. Upon substituting $v = \Sigma a(k) u_k$ in (26) we find:

$$\Sigma a(k)\, \lambda(k)\, u_k + \frac{\alpha\tau}{2}\, e^{-j\alpha\tau}\, (m(-1)\, u_{-1} - m(1)\, u_1)$$
$$= \lambda(0)\, \Sigma a(k)\, u_k \tag{27}$$

Here we used the definition of A and B as given by (9), (10) and (13). Equating the coefficients of the u_k in the two sides of (27), we conclude that:

$$a(k) = 0 \quad \text{for} \quad k \neq 0, \pm 1$$
$$a(-1) = \frac{m(-1)\, e^{-j\alpha\tau}}{\lambda(0) - \lambda(-1)}\, \frac{\alpha\tau}{2}$$
$$a(1) = -\frac{m(1)\, e^{-j\alpha\tau}}{\lambda(0) - \lambda(1)}\, \frac{\alpha\tau}{2}$$

The value of $a(0)$ is arbitrary (this phenomenon always occurs in perturbation calculus, cf. reference 8, and can be traced back to the fact that an eigenvector of a linear operator belonging to a particular eigenvalue is only determined up to a scalar factor). The most convenient choice for us is $a(0) = 0$.

Next we turn our attention to equation (20), rewritten as:

$$Aw + Bv + Cu_0 = \lambda(0)\, w + \mu u_0 \tag{28}$$

As before we have $w = \Sigma b(k)\, u_k$; in view of the definition of C as given by (9) and (14), a substitution yields:

$$\Sigma b(k)\, \lambda(k)\, u_k + \Bigg\{ a(-1)\, m(-2)\, e^{-j(\alpha - \omega_m)\tau} \times \frac{(\alpha - \omega_m)\,\tau}{2}\, u_{-2} + m(0) \times \left[-a(-1)\, e^{-j(\alpha - \omega_m)\tau}\, \frac{(\alpha - \omega_m)\,\tau}{2} + a(1)\, e^{-j(\alpha + \omega_m)\tau}\, \frac{(\alpha + \omega_m)\,\tau}{2} \right] u_0$$
$$- a(1)\, m(2)\, e^{-j(\alpha + \omega_m)\tau}\, \frac{(\alpha + \omega_m)\,\tau}{2}\, u_2 \Bigg\}$$
$$+ \left\{ \frac{(\alpha\tau)^2}{8}\, e^{-j\alpha\tau} [m(-2)\, u_{-2} + 2m(0)\, u_0 + m(2)\, u_2] \right\}$$
$$= \lambda(0)\, \Sigma b(k)\, u_k + \mu u_0$$

Equating the coefficients of u_0 in the two members, we obtain the following expression for the second-order term in the power series expansion (18) of $\tilde{\lambda}(\alpha, \epsilon)$ in the variable ϵ:

$$\mu = m(0)\, e^{-j\alpha\tau} \left[a(1)\, \frac{(\alpha + \omega_m)\,\tau}{2}\, e^{-j\omega_m\tau} - a(-1)\, \frac{(\alpha - \omega_m)\,\tau}{2}\, e^{j\omega_m\tau} - \frac{(\alpha\tau)^2}{4} \right] \tag{29}$$

Considering the coefficients of u_k for $k \neq 0$ yields:

$$b(k) = 0 \quad \text{for} \quad k \neq 0, \pm 2$$
$$b(-2) = \frac{m(-2)\, e^{-j\alpha\tau}}{\lambda(0) - \lambda(-2)} \times \left[a(-1)\, \frac{(\alpha - \omega_m)\,\tau}{2}\, e^{j\omega_m\tau} + \frac{(\alpha\tau)^2}{8} \right]$$
$$b(2) = \frac{m(2)\, e^{-j\alpha\tau}}{\lambda(0) - \lambda(2)} \times \left[-a(1)\, \frac{(\alpha + \omega_m)\,\tau}{2}\, e^{-j\omega_m\tau} + \frac{(\alpha\tau)^2}{8} \right]$$

Hence w is determined up to its u_0-component, and again we choose $b(0) = 0$.

Treating equations (21) and (22) in the same way (now using (15) and (16)), we are able to compute the fourth order term:

$$\nu = m(0)\, e^{-j\alpha\tau} \Bigg[c(1)\, \frac{(\alpha + \omega_m)\,\tau}{2}\, e^{-j\omega_m\tau} - c(-1)\, \frac{(\alpha - \omega_m)\,\tau}{2}\, e^{j\omega_m\tau} + b(2)\, \frac{(\alpha + 2\omega_m)^2\,\tau^2}{8}\, e^{-2j\omega_m\tau} + b(-2)\, \frac{(\alpha - 2\omega_m)^2\,\tau^2}{8}\, e^{2j\omega_m\tau}$$
$$- a(1)\, \frac{(\alpha + \omega_m)^3\,\tau^3}{16}\, e^{-j\omega_m\tau} + a(-1)\, \frac{(\alpha - \omega_m)^3\,\tau^3}{16}\, e^{j\omega_m\tau} + \frac{(\alpha\tau)^4}{64} \Bigg] \tag{30}$$

The numbers $c(1)$ and $c(-1)$ appear in the expansion $z = \Sigma c(k)\, u_k$ and are given by:

$$c(1) = \frac{m(1)\, e^{-j\alpha\tau}}{\lambda(0) - \lambda(1)} \left[b(2)\, \frac{(\alpha + 2\omega_m)\,\tau}{2}\, e^{-2j\omega_m\tau} - a(1)\, \frac{(\alpha + \omega_m)^2\,\tau^2}{4}\, e^{-j\omega_m\tau} + a(-1)\, \frac{(\alpha - \omega_m)^2\,\tau^2}{8}\, e^{j\omega_m\tau} + \frac{(\alpha\tau)^3}{16} \right]$$

$$-\frac{a(1)\,\mu}{\lambda(0)-\lambda(1)}$$

$$c(-1)=\frac{m(-1)\,e^{-j\alpha\tau}}{\lambda(0)-\lambda(-1)}$$

$$\times\left[-b(-2)\,\frac{(\alpha-2\omega_m)\,\tau}{2}\,e^{2j\omega_m\tau}\right.$$

$$-a(-1)\,\frac{(\alpha-\omega_m)^2\,\tau^2}{4}\,e^{j\omega_m\tau}$$

$$\left.+\,a(1)\,\frac{(\alpha+\omega_m)^2\,\tau^2}{8}\,e^{-j\omega_m\tau}-\frac{(\alpha\tau)^3}{16}\right]$$

$$-\frac{a(-1)\,\mu}{\lambda(0)-\lambda(-1)}$$

It should be clear by now how higher-order terms can be computed but also that the computations get very involved.

Numerical results and comparison with experiment

The experiments reported in the sequel were performed on a SENNA lathe (15 kW) provided with a special toolholder so as to come as close to the theoretical one-degree-of-freedom assumption as possible. Its characteristics were $\omega_0/2\pi$ = 189 Hz and ζ = 0.04. Further comments, where the computer programs used to obtain the numerical results are described in full detail as well, are given by van Bogaert.[9]

Spectral analysis of chatter under variable spindle speed

From the reasoning given earlier in the paper it follows that the chatter vibration under varying spindle speed is a linear combination of pure vibrations with frequencies $\alpha/2\pi$, $\alpha \pm \omega_m/2\pi$, $\alpha \pm 2\omega_m/2\pi$, etc. where the 'central' frequency $\alpha/2\pi$ is near to the corresponding frequency $\alpha_0/2\pi$ of chatter under constant speed, and $\omega_m/2\pi$ is the frequency of the speed modulation.

An unexpected and striking experimental verification of this fact could be obtained by Fourier-analysing the chatter under *a priori* constant speed. Although in principle only one frequency should occur, the result (*Figure 1*) shows several frequencies, organized around a central frequency of 209.5 Hz in an equidistant way. The distance between successive peaks equals 4.75 Hz, which corresponds exactly to the spindle speed of 285 rpm used for the experiment. Hence the frequency pattern in *Figure 1* can be understood by referring to the slight speed modulation caused by the imbalance of the workpiece itself (the amplitude of the modulation was estimated 0.5 rpm).

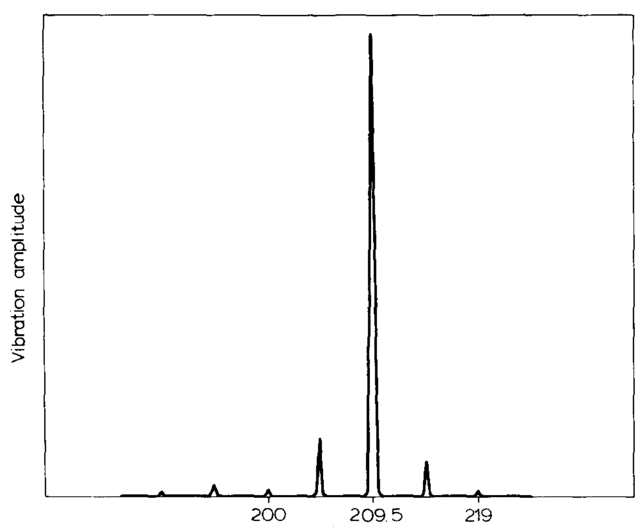

Figure 1 Fourier analysis of chatter vibration for constant lag case. Sidebands are due to periodic fluctuations in rotational speed. Average spindle speed, 285 rpm; ω_m = 4.75 X 2π rad/s

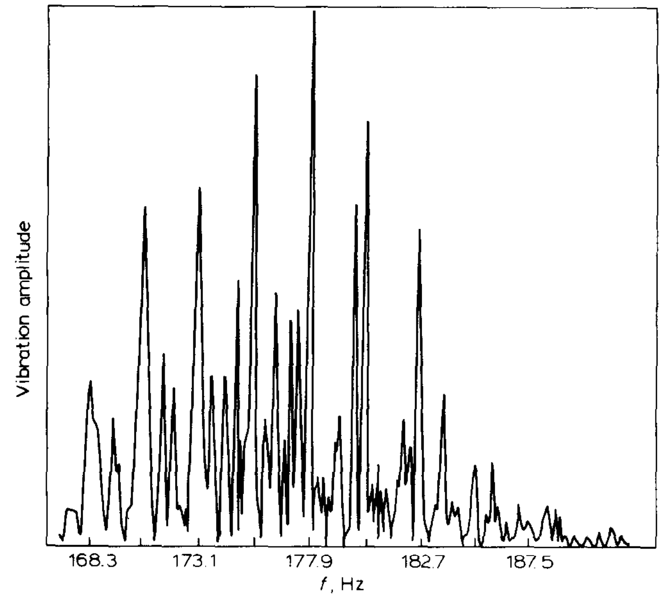

Figure 2 Fourier analysis of chatter vibration under speed modulation. Complexity of spectrum is due to interference of two speed fluctuations. Average spindle speed, 290 rpm; ω_{m1} = 0.4 X 2π rad/s; ω_{m2} = 4.8 X 2π rad/s

It is not surprising, then, that the spectrum under (externally imposed) speed modulation (*Figure 2*) turns out to be more complicated than expected on theoretical grounds, since it arises from a superposition of two modulations. Still, the fine structure shows evenly spaced frequencies approximately 0.4 Hz apart in accordance with the speed modulation frequency. The seemingly dominant frequencies at distance of 2.4 Hz must be the result of interference with the 'intrinsic modulation' of 4.8 Hz corresponding to the average spindle speed of 290 rpm.

Influence of the amplitude of the modulation

It is clear that the theory developed in the previous sections is especially useful to study the behaviour of the critical depth of cut (or, equivalently $K_m(T)/K_c$) as a function of the amplitude T of the time-lag modulation using the procedure outlined at the end of the previous section. A graph for a particular case is depicted in *Figure 3*; the amplitude of speed modulation v_{var} is used as a variable rather than T to render the comparison with practice easier. The results of both second-order approximation (using only μ, (29)) and fourth-order approximation (taking ν, (30), into account as well) are shown. Although even fourth-order approximation ceases to be meaningful for relatively small modulation amplitudes, it nevertheless suggests that modulation with an amplitude of 20 rpm (vs. a mean speed of 290 rpm) already allows for a 50% improvement of cutting depth.

These results are in reasonable agreement, for small variation amplitudes, with those obtained experimentally in the study,[9] part of which are summarized in *Figure 4*. For

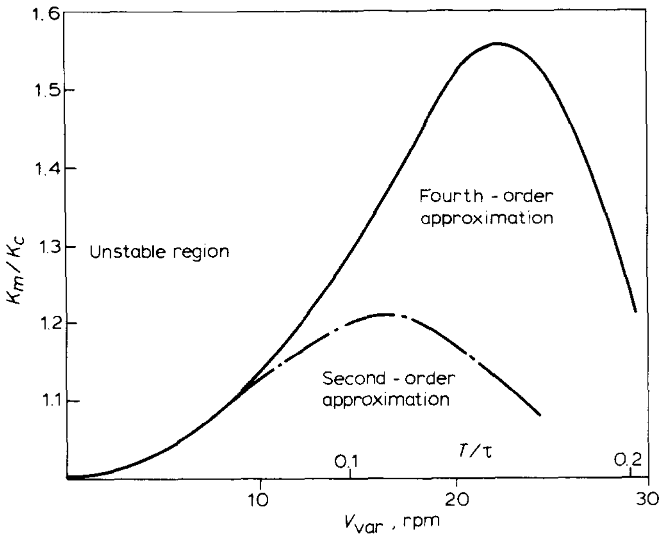

Figure 3 Theoretical behaviour of critical depth of cut in function of speed modulation amplitude v_{var} for second and fourth-order approximation. Average spindle speed, 290 rpm; mm = 0.5 X 2π rad/s; f_0 = 189 Hz; ζ = 0.04

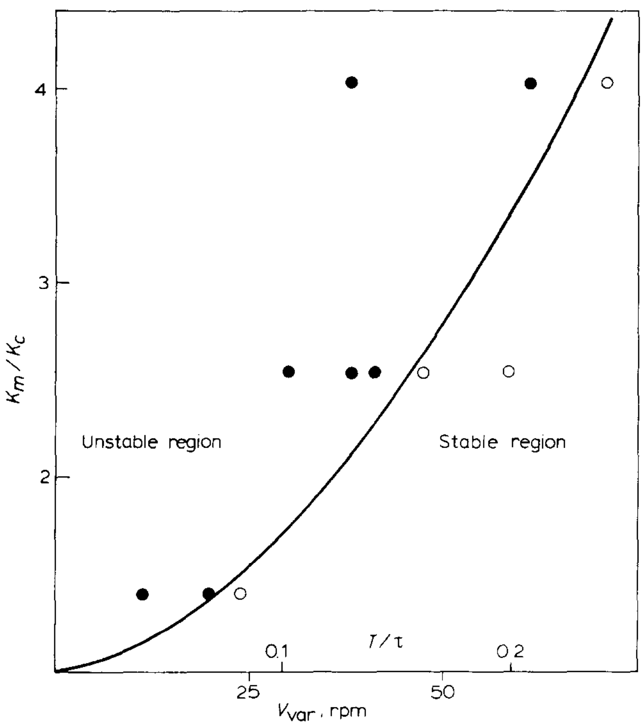

Figure 4 Experimental results showing increase of critical depth of cut with increasing speed modulation amplitude v_{var}. Average spindle speed, 290 rpm; ω_m = 0.5 X 2π rad/s; (o), stable operation point; (•), unstable operation point

the methodology used to collect these data, particularly for the definition of stability, see van Bogaert.[9]

In order to check numerically a larger portion of the curve in *Figure 4* (for greater values of T) it would be necessary to use sixth- or eighth-order approximation in the above perturbation calculus.

In general the convergence of the perturbation series (18) was observed to depend on the values of the parameters ω_0 and ζ: it improves with increasing ζ and decreasing ω_0.

Stability diagram

For comparison with Sexton's results,[5] we computed the stability diagram for the values $\omega_0/2\pi$ = 50 Hz, ζ = 0.05,

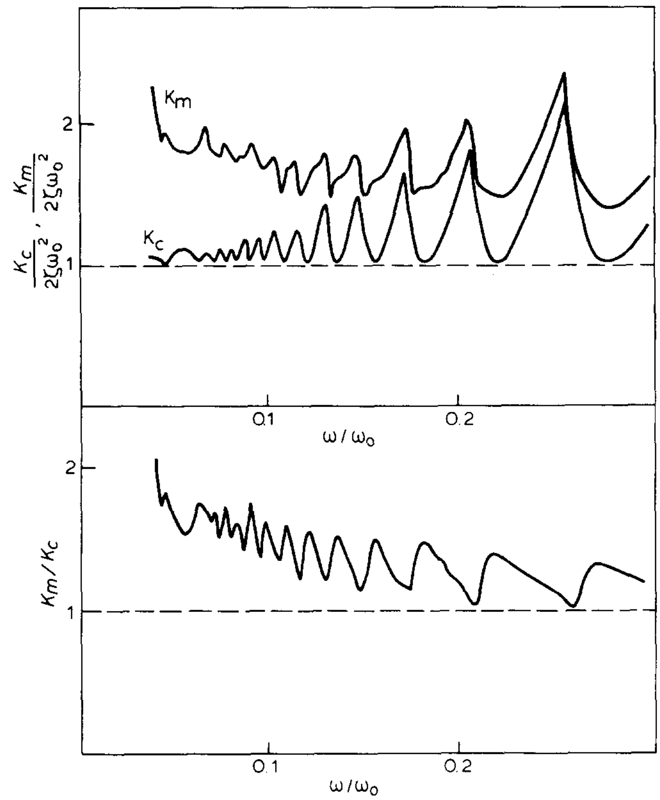

Figure 5 Stability diagram and relative efficiency K_m/K_c computed for same characteristics as Sexton *et al.*[5] ω_0 = 50 X 2π rad/s; ζ = 0.05; ω_m = 0.5 X 2π rad/s; T/τ = 0.1

$\omega_m/2\pi$ = 0.5 Hz, and for a speed variation of 10% of the average spindle speed (*Figure 5*; the relative efficiency K_m/K_c is also given). With these characteristics, the fourth order approximation was found to give meaningful results for T/τ up to 0.1, in favourable contrast with situation described in *Figure 3*.

Not surprisingly, the results are qualitatively close to Sexton's: they show an overall increase in stability, which is more important for lower values of $\omega = 2\pi/\tau$ (actually for other characteristics occasional decreases in stability were exhibited for some larger values of ω). A direct quantitiative comparison with Sexton's numerical figures (*Figure 4* is reference 5) is not strictly possible because they use T/τ = 0.2 which is definitely beyond the limit of validity of our approximation for the given values of ω_0 and ξ. Still, our estimates (for T/τ = 0.1) of the increase in stability are higher than Sexton's for T/τ = 0.2.

Since K_m/K_c can be expected to increase with T/τ, this difference is significant.

Conclusion

Using the techniques of perturbation calculus, we constructed a non-periodic approximate solution to equation (1) describing cutting under modulated spindle speed, thus removing the nonphysical assumption that the chatter vibration under these circumstances be approximately periodic, which was essential in earlier work.[4,5] A clear insight in the structure of chatter has been obtained and it has been shown in detail to what extent increasing the amplitude of the speed fluctuation improves the stability. For small amplitudes the computed values are very close to the observed ones.

Our study of stability based on a fourth-order perturbation series in terms of the amplitude of the speed variation is in essential qualitative agreement with Sexton *et al.*,[5] but it suggests a more substantial possible increase in stability, in better accordance with the experiments. Thus the analysis presented here is likely to provide a helpful tool to develop variable spindle speed machining for the purpose of avoiding chatter, although it may be necessary to take extra terms of the perturbation series into account in order to collect numerical data for a larger range of possible machine characteristics and speed modulation amplitudes.

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